Modern approaches to quantum gravity

Homework 2 Fall 2025

1. Mechanics of spherically symmetric black holes

In this exercise we will verify some special cases of the laws of black hole mechanics for the symmetrical black hole

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \tag{1}$$

characterised by general functions $f(r)$ that has a simple zero at the horizon $r = r_h$. We will restrict to asymptotically flat black holes so that $f(r) \xrightarrow{r\to\infty} 1$.

(a) Consider the Euclidean geometry obtained by Wick rotating $t = i\tau$. By studying the geometry near $r = r_h$ with an appropriate coordinate change, show that it has no conical singularity if τ has periodicity β given by

$$\beta = \frac{4\pi}{|f'(r_h)|}\,. \tag{2}$$

Note: The periodicity can be interpreted as the inverse temperature of a canonical ensemble, because if one considers the partition function of a gravitational system in the semiclassical limit, one obtains:

$$Z_{\text{grav}}[\beta] \approx e^{-S^{\beta}_{\text{grav}}[g_{\text{classical}}]} \tag{3}$$

where $S^{\beta}_{\text{grav}}[g_{\text{classical}}]$ is the Euclidean gravitational action evaluated on solution of the Einstein equations with periodic boundary conditions, implying $\tau \sim \tau + \beta$. Thus, the temperature is related to the parameters of the gravitational solution. If one varies the temperature, the geometry will adjust itself to match the required boundary conditions.

(b) Explain the general relation between surface gravity and temperature of a black hole, and show that the temperature found earlier agrees with the surface gravity obtained through the formula

$$\xi^b \nabla_b \xi_a|_{r_h} = -\frac{1}{2}\nabla_a(\xi^b\xi_b)|_{r_h} = \kappa \xi_a|_{r_h} \tag{4}$$

Hint: Consider the Eddington-Finkelstein coordinates (v, r, θ, ϕ), where $dv = dt + \frac{dr}{f(r)}$, which are not singular on the horizon, and employ the appropriately normalised null Killing vector ξ^a.

(c) Find the temperature T of the Reissner-Nordstrom black hole with mass M and charge Q, described by

$$f(r) = 1 - \frac{2MG}{r} + \frac{Q^2 G}{r^2} \tag{5}$$

(d) Find the area of the black hole horizon $A(M,Q)$ as a function of M, Q, and determine the potential Φ associated to the electric charge, such that the first law is satisfied:

$$\frac{1}{4G}T\delta A = dM - \Phi\delta Q \tag{6}$$

(e) Consider the collision process of two Schwarzschild black holes with masses M_1 and M_2, which are separated by a large distance $d \gg M_1, M_2$. Assume that after some complicated dynamical evolution, the final state is a Schwarzschild black hole with mass M_3. You can neglect quantum effects and work classically, so the area theorem applies.[1] In this case, what is the minimum allowed value for the mass M_3 of the final black hole ?

2. Raychaudhuri equation

(a) Assume matter satisfies the null energy condition $T_{\mu\nu}k^\mu k^\nu \geq 0 \quad \forall k$ null. Show that if $\theta = \theta_0 < 0$ at any point along a null geodesic, then $\theta \to -\infty$ along that geodesic within finite affine parameter $\lambda \leq 2/|\theta_0|$, assuming that $\hat{\omega}_{\mu\nu} = 0$.

(b) We will now verify the Raychaudhuri equation in the case of a Schwarzchild black hole. It is useful to describe it in the Kruskal coordinates (we set $G = 1$),

$$ds^2 = -\frac{32M^3 e^{-r/(2M)}}{r} dU dV + r^2 d\Omega^2 \,, \tag{7}$$

where the function $r = r(U,V)$ is implicitly defined through the relation

$$UV = e^{r/(2M)}\left(1 - \frac{r}{2M}\right) . \tag{8}$$

Remember that the black hole interior corresponds to the region $U > 0$, $V > 0$. Let us consider null outgoing geodesics behind the horizon. The advantage of Kruskal coordinates is that the outgoing null motion inside the black hole corresponds to fixed $U > 0$ and varying $V \to \infty$. Show that the null, affine parameterized geodesics x^μ have a tangent vector

$$\frac{dx^\mu}{d\lambda} = \xi^\mu, \qquad \xi^\mu = r e^{r/(2M)} \left(\frac{\partial}{\partial V}\right)^\mu . \tag{9}$$

Hint: Check the affine parameterized geodesic equation, $\xi^\alpha \nabla_\alpha \xi^\mu = 0$.

(c) (Optional) Compute the expansion parameter θ for these geodesics inside/outside the Schwarzschild horizon.

Hint: Use the property $\nabla_\mu \xi^\mu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\xi^\mu)$

You should find

$$\theta = -\frac{8M^2}{r} U \,. \tag{10}$$

[1] When quantum effects are taken into account, only the generalized second law applies. In principle, the final state's entropy may be dominated by radiation rather than by the black hole's area.

(d) Using the above result, show that these geodesics satisfy the Raychaudhuri equation with no shear and no twist,

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 . \tag{11}$$

Hint: Use the chain rule $\frac{d}{d\lambda} = \frac{dV}{d\lambda}\frac{d}{dV}$.

Argue why the shear and twist do not appear.

(e) You proved in point (a) that, when $\theta = \theta_0 < 0$ at any point along a null geodesic, $\theta \to -\infty$ along that geodesic within finite affine parameter $\lambda_0 \leq 2/|\theta_0|$. What is λ_0 for the null geodesic we considered ?